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ON AN IDEA OF CHETAEV

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We develop an idea of Chetaev to the effect that if the measurement errors are sufficiently small, we can find the best conditions for the experimental investigation of a material object, using the property of instability of its mathematical model [1]. We show that in a number of cases we can significantly weaken the requirements on the accuracy of measurements when experimentally seeking new properties of the object, other than the properties of its known mathematical model. To do this it is sufficient to use a property of the mathematical model stronger than instability. In the paper we call this property the property of complete lability. We have found the sufficient conditions for complete lability. We introduce the notion of the degree of lability. We consider the problem of controlling the degree of lability. We apply the results obtained to two systems: the mathematical model of the struggle for existence between two biological species and the problem of power-off entry of a spacecraft into a planetary atmosphere (passive descent).

1. We consider a class of objects whose motion can be described by the system of differential equations

$$dx/dt = X(t, x), \qquad t_0 \leqslant t < t_* \tag{1.1}$$

Here $x = (x_1, \ldots, x_n)$ is a vector in a real n-dimensional linear normed space R_x^n with norm $\|x\| = \max_i |x_i|$, t is the time, t_0 is the initial instant, $t_0 \in (-\infty, \infty)$, t_* is either a number on the halfline $t > t_0$ or the symbol ∞.

Let V_0 be a given set in R_x^n of the initial states $x(t_0) = x_0$; $x(t) = f(t, x_0, t_0)$, $t_0 \leqslant t < t_*$, $x_0 \in V_0$ is a solution (a trajectory) of system (1.1) in $R_x^n \times I$, where $I = \{t : t_0 \leqslant t < t_*\}$. We assume that $X(t, x)$ satisfies the conditions which ensure the existence and continuity of the function $f(t, x_0, t_0)$ for all $t \in I$ and for all $x_0 \in V_0$.

Following Chetaev [2] we shall "regard a concrete phenomenon as a theoretical phenomenon perturbed by small forces not fully accounted for and by deviations of the initial conditions". Furthermore, following the recommendations in [1] we delimit the structure of the perturbing forces: we assume that the difference of the material object's behavior from that of its model (1.1) is caused only by the difference in their initial states. Consequently, if $f(t, x_0, t_0)$, $t_0 \leqslant t < t_*$ is a theoretical trajectory, the trajectory of the object's model (1.1), then $f(t, x_0^*, t_0)$, $t_0 \leqslant t < t_*$ is the true trajectory of the object, while $\Delta x_0 = x_0^* - x_0$ is the perturbations, not fully accounted for, of the initial state of the model, equal to the deviation of the object's true initial state x_0^* from the theoretical initial state, i.e. the initial state $x(t_0) = x_0$ of its model (1.1).

It is clear that when arbitrarily small deviations Δx_0 of the object's initial state

x_0^* from the initial state x_0 of its model (1.1) lead to an arbitrarily large increase of the norm of the difference $\| f(t, x_0^*, t_0) - f(t, x_0, t_0)\|$ as time t goes on, the difference between the object's behavior and that of its model can be detected even by crude imprecise observation means. In this case favorable conditions arise for the experimental detection of those properties of the object which differ from the properties of its known model (1.1), including its new unexpected properties. This arbitrarily strong subjection of model (1.1) to small perturbations of its initial state x_0 is a stronger property than the Liapunov instability of its trajectories, since here we are dealing with an unbounded increase in the distance between the states of system (1.1) in R_x^n on the unperturbed and on the perturbed trajectories.

Definition. A trajectory $C = \{f(t, x_0, t_0),\ t_0 \leqslant t < t_*,\ x_0 \in V_0\}$ is said to be completely labile relative to set V_0 if for any arbitrarily large number $\varepsilon > 0$, and for any arbitrarily small number $\delta > 0$, among the set of trajectories C there exist at least two trajectories

$$f(t, (x_0)_1, t_0),\ f(t, (x_0)_2, t_0),\ t_0 \leqslant t < t_*;\ (x_0)_1, (x_0)_2 \in V_0$$

and an instant $t_1 = t_1(\varepsilon, \delta, (x_0)_1, (x_0)_2)$, $t_0 < t_1 < t_*$, such that

a) $0 < \| (x_0)_1 - (x_0)_2 \| \leqslant \delta$, (b) $\| f(t_1, (x_0)_1, t_0) - f(t_1, (x_0)_2, t_0) \| > \varepsilon$.

Here and later t_* either is a number on the halfline $t > t_0$ or is the symbol ∞.

Theorem 1. Let the right-hand side $X(t, x)$ of system (1.1) be a function of t and x, continuous together with its partial derivatives with respect to t and x in $R_x^n \times I$, where $I = \{t : t_0 \leqslant t < t_*\}$. We denote

$$L(t, x_0, t_0) = \int_{t_0}^{t} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} X_i(\tau, f(\tau, x_0, t_0))\, d\tau \tag{1.2}$$

Then in order for trajectories C to be completely labile relative to set V_0, it is sufficient that

$$L(t, x_0, t_0) \to \infty \quad \text{as} \quad t \to t_* \tag{1.3}$$

uniformly with respect to $x_0 \in V_0$.

Proof. Let $\bar{x}_0$ be any point from V_0. From the theorems on existence, uniqueness and continuous dependence of trajectories C on x_0, it follows that the mapping $T_t : x = f(t, x_0, t_0)$, $x_0 \in V_0$, is continuous and one-to-one for every t, $t_0 \leqslant t < t_*$. By hypothesis V_0 is an open set, therefore, there exists a set of spheres

$$S_{\delta_0}(\bar{x}_0) = \{x_0 : \| x_0 - \bar{x}_0 \| \leqslant {}^1/_2\, \delta_0\}, \quad S_{\delta_0}(\bar{x}_0) \subset V_0$$

where $\delta_0 > 0$ is any number not exceeding a specified number $\delta > 0$ no matter how small the latter may be. The continuous function $f(t, x_0, t_0)$, $x_0 \in S_{\delta_0}(\bar{x}_0)$, defines the mapping T_t of the sphere $S_{\delta_0}(\bar{x}_0)$ onto the set $S_{\rho_t} = T_t(S_{\delta_0}(\bar{x}_0))$, where S_{ρ_t} is a closed connected set (a connected compactum), bounded for each fixed t, $t_0 \leqslant t < t_*$, whose diameter equals ρ_t.

We denote the Jacobian

$$J(t, x) = \frac{\partial(x_1, \ldots, x_n)}{\partial(x_{10}, \ldots, x_{n0})}$$

According to Liouville's theorem [3]

$$J(t, f(t, x_0, t_0)) = \exp \int_{t_0}^{t} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} X_i(\tau, f(\tau, x_0, t_0))\, d\tau \tag{1.4}$$

Therefore, $J(t, f(t, x_0, t_0)) \neq 0$. Let $|S_{\rho_t}|$ be the volume of set S_{ρ_t}. Taking the continuity of $J(t, f(t, x_0, t_0))$ on x_0 into account, we obtain

$$|S_{\rho_t}| = \int\limits_{S_{\rho_t}} dx = \int\limits_{S_{\delta_0}(\bar{x}_0)} J(t, f(t, x_0, t_0))dx_0 = |S_{\delta_0}(x_0)|\, J(t, f(t, a, t_0)) \tag{1.5}$$

Here $a = a(t)$ is some point from $S_{\delta_0}(x_0)$, fixed for a fixed t, $t_0 \leqslant t < t_*$. From (1.4), (1.5) it follows that

$$|S_{\rho_t}| = |S_{\delta_0}(x_0)| \exp \int\limits_{t_0}^{t} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} X_i(\tau, f(\tau, a, t_0))\, d\tau \tag{1.6}$$

The diameter ρ_t of compactum S_{ρ_t} is

$$\rho_t = \sup_{x, x' \in S_{\rho_t}} \|x - x'\| = \|x^1 - x^2\| \tag{1.7}$$

where x^1 and x^2 belong to the boundary $\Gamma(S_{\rho_t})$ of set S_{ρ_t} by virtue of the continuity of the norm $\|x - x'\|$. The preimages of the points x^1, x^2

$$(x_0)_1 = T_t^{-1}(x^1), \qquad (x_0)_2 = T_t^{-1}(x^2)$$

where T_t^{-1} is a continuous mapping, inverse to T_t, also belong to the boundary $\Gamma(S_{\delta_0}(x_0))$ of the sphere $S_{\delta_0}(x_0)$. Therefore

$$\|(x_0)_1 - (x_0)_2\| \leqslant \delta_0 \leqslant \delta \tag{1.8}$$

Note that

$$|S_{\rho_t}| \leqslant (\rho_t)^n \tag{1.9}$$

With due regard to notation (1.2), from (1.6), (1.9) we obtain the estimate

$$\rho_t \geqslant |S_{\delta_0}(x_0)|^{1/n} \exp[L(t, a, t_0)/n], \quad a \in S_{\delta_0}(x_0) \subset V_0 \tag{1.10}$$

The theorem's assertion follows from inequality (1.10) and condition (1.3).

2. We introduce the function of positive terms

$$h(t, \delta_0, x_0, t_0) = \left(\frac{\pi}{2}\right)^{(n-\varkappa)/2n} (n!!)^{-1/n} \delta_0 \exp[L(t, x_0, t_0)/n]$$

$$\varkappa = \begin{cases} 0, & n = 2, 4, 6, \dots \\ 1, & n = 3, 5, 7, \dots \end{cases} \tag{2.1}$$

Here $n!!$ is the product of all positive integers of the same parity as n but not exceeding n.

Theorem 2. Let the right-hand side $X(t, x)$ of system (1.1) be a function of t and x, continuous together with its derivatives with respect to t and x in $R_x^n \times I$. Then the trajectories C are completely labile relative to set V_0 if for any (arbitrarily large) number $\varepsilon > 0$ and for any (arbitrarily small) number $\delta > 0$ there exists: (a) an initial state $x_0 \in V_0$, (b) a sphere $S_{\delta_0}(x_0) \subset V_0$ of diameter $\delta_0 \leqslant \delta$, (c) an instant $t_1 = t_1(\varepsilon, \delta_0)$, $t_0 < t_1 < t_*$, such that

$$h(t_1, \delta_0, b, t_0) \geqslant \varepsilon \quad \text{for each point } b \in S_{\delta_0}(x_0) \tag{2.2}$$

Proof. Let us return to (1.10), where the volume $|S_{\delta_0}(x_0)|$ of the sphere $S_{\delta_0}(x_0)$ of diameter δ_0 can be computed by Jacobi's formula [4]; as a result (1.10) takes the form

$$\rho_t \geqslant h(t, \delta_0, a, t_0), \qquad a \in S_{\delta_0}(\bar{x}_0) \tag{2.3}$$

Then it is easy to complete the proof of the theorem by keeping (2.2), as well as (1.7), (1.8) and (2.3) in mind.

Note 1. The quantity ρ_t on the left-hand side of (2.3) characterizes the error in the determination of the state of system (1.1) at the instant t, caused by the error δ_0 with which its initial state is specified, while $h(t, \delta_0, a, t_0)$ in (2.3) characterizes the minimum level of this error.

Note 2. The use of inequality (2.2) for the detection of completely labile trajectoties of system (1.1) presupposes that the function $L(t, x_0, t_0)$ either can be computed exactly or is effectively bounded from below.

3. Let us consider the mathematical model of the struggle for existence between two biological species [5], taken as the original one in [6] and relating to a broad class of so-called models of evolution [7]

$$\frac{dx_1}{dt} = x_1(\varepsilon_1 - \gamma_1 x_2), \qquad \frac{dx_2}{dt} = -x_2(\varepsilon_2 - \gamma_2 x_1), \qquad t \geqslant 0 \tag{3.1}$$

Here $\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2$ are positive numbers; x_1, x_2 are the coordinates of the vector $x = (x_1, x_2) \in R_x^2$, $x_0 = (x_{10}, x_{20})$ is the state of system (3.1) at the instant $t = 0$; $V_0 = \{x_0 : 0 < x_{10}, x_{20} < \beta\}$, $\beta > 0$ is a number.

Let us show that the trajectories of system (3.1), starting off on V_0 at $t = 0$ are completely labile relative to V_0. In order to make use of Theorem 2 it is necessary to find the quantity

$$L(t, x_0, 0) = \int_0^t (\varepsilon_1 - \gamma_1 x_2(\tau) - \varepsilon_2 + \gamma_2 x_1(\tau))\, d\tau \tag{3.2}$$

$$x_1(t) = f_1(t, x_0, 0), \; x_2(t) = f_2(t, x_0, 0), \; t \geqslant 0, \; x_0 \in V_0$$

Here $x_1(t)$, $x_2(t)$ is the solution of the system (3.1). With due regard to (3.1), the integrand in (3.2) can be represented as

$$(\varepsilon_1 - \gamma_1 x_2(t) - \varepsilon_2 + \gamma_2 x_1(t))\, dt = dx_1/x_1 + dx_2/x_2 \tag{3.3}$$

Substitution of (3.3) into (3.2) yields

$$L(t, x_0, 0) = \ln\left|\frac{x_1(t)\, x_2(t)}{x_{10} x_{20}}\right| \tag{3.4}$$

On the phase plane R_x^2 we construct a circle of diameter $\delta_0 \leqslant \beta$

$$S_{\delta_0}(\bar{x}_0) = \left\{x_0 : \left(x_{10} - \frac{m+1}{2m}\alpha\right)^2 + \left(x_{20} - \frac{m+1}{2m}\alpha\right)^2 \leqslant \left(\frac{m-1}{2lm}\alpha\right)^2\right\} \tag{3.5}$$

$$l \geqslant 1, \; m > 1, \; 0 < \alpha \leqslant lm\beta / (m-1)$$

Using (3.4), from formula (2.1) for $n = 2$ we find

$$h(t, \delta_0, x_0, 0) = 0.8862269\ldots \frac{m-1}{lm}\alpha \left|\frac{f_1(t, x_0, 0)\, f_2(t, x_0, 0)}{x_{10} x_{20}}\right| \tag{3.6}$$

$$x_0 \in S_{\delta_0}(\bar{x}_0)$$

Let $b = (b_1, b_2)$ be any point of $S_{\delta_0}(\bar{x}_0)$. From (3.5) it follows that

$$b_1 b_2 \leqslant \left(\frac{m-1}{m}\alpha\right)^2 \tag{3.7}$$

The number $\alpha > 0$ in (3.5) can be chosen so small that on the plane R_x^2 the singular point $(\varepsilon_2 / \gamma_2, \varepsilon_1 / \gamma_1) \notin S_{\delta_0}(\bar{x}_0)$.

It is known from [5] that Eqs. (3.1) have a unique singular point $(\varepsilon_2 / \gamma_2, \varepsilon_1 / \gamma_1)$ in the quadrant $x_1 > 0, x_2 > 0$, which is a center, the phase curves of system (3.1) in this quadrant are all closed, and the time variation of the phase coordinates x_1, x_2 takes place by a periodic law. Consequently, all the phase curves of system (3.1) should surround the point $(\varepsilon_2 / \gamma_2, \varepsilon_1 / \gamma_1)$ in the quadrant $x_1 > 0, x_2 > 0$ (otherwise, by the Bendixson theorem [8] there would be a singular point other than $(\varepsilon_2 / \gamma_2, \varepsilon_1 / \gamma_1)$ in this quadrant). Therefore, for each point $x_0 \in S_{\delta_0}(\bar{x}_0)$ there exists an instant $t_1 = t_1(x_0)$, $0 < t_1 < \infty$, such that

$$f_1(t_1, x_0, 0) > \varepsilon_2 / \gamma_2, \qquad f_2(t_1, x_0, 0) > \varepsilon_1 / \gamma_1 \tag{3.8}$$

The functions $f_1(t, x_0, t_0)$, $f_2(t, x_0, t_0)$ depend continuously on $x_0 \in V_0$, therefore, for any fixed $\alpha > 0$ there exists an integer $l_0 = l_0(\alpha) \geqslant 1$, depending on α, such that for any circle (3.5), where $l \geqslant l_0 = l_0(\alpha)$ the inequalities (3.8) are fulfilled for any point $x_0 = b = (b_1, b_2) \in S_{\delta_0}(\bar{x}_0)$. Consequently, from (3.8) we have

$$f_1(t_1, b, 0) f_2(t_1, b, 0) > \varepsilon_1\varepsilon_2/\gamma_1\gamma_2 \tag{3.9}$$

Then, with due regard to (3.7), (3.9), from (3.6) we obtain

$$h(t_1, \delta_0, b, 0) \geqslant 0.8862269\ldots \frac{m\varepsilon_1\varepsilon_2}{(m-1)\gamma_1\gamma_2} \frac{1}{\alpha}$$

Consequently, whatever the number $\varepsilon > 0$, by choosing $\alpha > 0$ sufficiently small we can always ensure the condition $h(t_1, \delta_0, b, t_0) \geqslant \varepsilon$ for any point $b = (b_1, b_2) \in S_{\delta_0}(\bar{x}_0)$. According to Theorem 2.1, this signifies that trajectories $C = \{(f_1(t, x_0, 0), f_2(t, x_0, 0)), t \geqslant 0, x_0 \in V_0\}$ are completely labile relative to V_0. It is necessary to note also that the diameter and the coordinates of the center of the circle (3.5) are arbitrarily small as $\alpha \to 0$, therefore, the origin of the phase plane R_x^2 is a point of condensation of those pairs of points $\{(x_0)_1, (x_0)_2\}$ the distance between which at the instant $t = 0$ is arbitrarily small, whereas the distance between the phase trajectories of system (3.1), issuing from each pair of these points, becomes larger than any $\varepsilon > 0$ as t increases.

4. Let all the conditions of Theorem 1 be fulfilled, excepting condition (1.3) which we do not take into account here. We fix a certain instant on $I = \{t: t_0 \leqslant t < t_*\}$. We consider the set $V_t = T_t(V_0)$, $V_0 \subset R_x^n$ and the sphere $S_{\delta_t}(\bar{x}) = \{x : \|x - \bar{x}\| \leqslant \frac{1}{2}\delta_t\}$ with center at any point $\bar{x} \in V_t$, where the number $\delta_t > 0$ is chosen such that $S_{\delta_t}(\bar{x}) \subset V_t$. By virtue of the assumptions made, for each point $x \in S_{\delta_t}(\bar{x})$ there exists a unique image $x_0 = T_t^{-1}(x)$, where $T_t^{-1} : x_0 = f(t_0, x, t)$, $t_0 \leqslant t < t_*$, $x \in V_t$. Let S_{δ_0} be the image of the sphere $S_{\delta_t}(\bar{x})$, $S_{\delta_0} = T_t^{-1}(S_{\delta_t}(\bar{x}))$, where δ_0 is the diameter of set S_{δ_0}. We denote

$$\mu(t, \bar{x}_0, t_0) = \lim_{\delta_t \to 0} (\delta_0 / \delta_t) \tag{4.1}$$

where $\bar{x}_0 = f(t_0, \bar{x}, t)$, δ_t is the diameter of sphere $S_{\delta_t}(\bar{x})$, δ_0 is the diameter of set S_{δ_0}. The quantity $\mu(t, \bar{x}_0, t_0)$ characterizes how strongly the trajectories of system (1.1), starting off in a neighborhood of point $\bar{x}_0$, condense i.e come together by the instant t. The larger the $\mu(t, \bar{x}_0, t_0)$, the higher the degree of condensability of the trajectories.

Let us find a lower bound for $\mu(t, x_0, t_0)$, where x_0 is any point of V_0. To do this, by analogy with (1.5) we can write

$$|S_{\delta_0}| = [J(t, f(t, a, t_0))]^{-1} |S_{\delta_t}(\bar{x})| \tag{4.2}$$

where $a = a(t)$ is some point of S_{δ_0}. If we take (1.2), (1.4) into consideration, then (4.2) can be rewritten as

$$|S_{\delta_0}| = |S_{\delta_t}(\bar{x})| \exp[-L(t, a, t_0)] \tag{4.3}$$

We keep in mind that $(\delta_0)^n \geqslant |S_{\delta_0}|$; we compute the volume $|S_{\delta_t}(\bar{x})|$ by Jacobi's formula [4]; then, with due regard to (4.1), from (4.3) we obtain the estimate

$$\mu(t, x_0, t_0) \geqslant \left(\frac{\pi}{2}\right)^{(n-\varkappa)/2n} (n!!)^{-1/n} \exp[-L(t, x_0, t_0)/n]$$

$$\varkappa = \begin{cases} 0, & n = 2, 4, 6, \ldots \\ 1, & n = 3, 5, 7, \ldots \end{cases} \tag{4.4}$$

From (4.4) it follows that if

$$L(t, x_0, t_0) \to -\infty \quad \text{as} \quad t \to t_* \tag{4.5}$$

then

$$\mu(t, x_0, t_0) \to \infty \quad \text{as} \quad t \to t_* \tag{4.6}$$

Consequently, one and the same quantity $L(t, x_0, t_0)$ characterizes, on the one hand, the complete lability of the trajectories of system (1.1), when $L(t, x_0, t_0) \to \infty$ (Theorems 1, 2), and on the other hand, the unrestricted condensability of the trajectories of system (1.1), when $L(t, x_0, t_0) \to -\infty$ (relations (4.5), (4.6)). Taking into consideration the singular role that the quantity $L(t, x_0, t_0)$ plays in all the preceding discussions, we will call the quantity $L(t, x_0, t_0)$ the degree of lability of system (1.1).

5. Let us consider the motion of a spacecraft at its power-off entry into a planetary atmosphere (passive descent). Under specified assumptions, the equations of planar motion of the spacecraft can be written as [9]

$$\frac{dH}{dz} = A(z, H)\,\theta, \qquad (v_0 - z)\frac{d\theta}{dz} = \frac{C_y}{C_x} - \frac{A(z, H)}{v_0 - z}\left[g - \frac{1}{r}(v_0 - z)^2\right]$$

$$A(z, H) = \left[\frac{C_x S \rho(0)}{2m_0}(v_0 - z)\, e^{-\lambda H}\right]^{-1}, \qquad 0 \leqslant z < v_0 - v \tag{5.1}$$

Here, in (obviously unessential) contrast to [9] we have chosen as the independent variable, instead of v, the scalar quantity $z = v_0 - v$, monotonously increasing with time, where $v_0 = v(t_0)$ is the velocity of the spacecraft at the initial instant $t = t_0$, $v = v(t)$ is the velocity of the spacecraft at the current instant t; the values of z and t are in one-to-one correspondence. As regards the other notation in (5.1), H is the altitude of the spacecraft above the planet's surface, θ is the angle of inclination of the craft's trajectory relative to the local horizon plane, λ is the logarithmic gradient of the atmosphere's density, S and m are the characteristic cross-sectional area and the mass of the craft, c_y and c_x are the lift and drag coefficients (*), respectively, $\rho(0)$ is the

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*) Editor's Note. See footnote on page 39.

atmosphere density at altitude $H = 0$, r is the planet's radius, g is the acceleration due to gravity.

The actual trajectories of the spacecraft can differ from the calculated ones for a number of reasons whose analysis leads to the dispersion problem. One of the reasons for the dispersion of the trajectories is the perturbation of the initial state of system (5.1), $H(t_0) = H_0$, $\theta(t_0) = \theta_0$. Let us estimate the condensability of the trajectories of system (5.1) with the aid of the numerical measure (4.1) of the condensability of the trajectories and of the relation (4.4). To do this we first compute the degree of lability of system (5.1) by formula (1.2). Keeping in mind the first equation in (5.1), the integral to be computed can be reduced to a quadrature, taken in finite form

$$L(z; H_0, \theta_0; 0) = \lambda \int_0^{v_0 - v} A(z, H(z))\,\theta(z)\,dz = \lambda \int_0^z dH(\tau) = \lambda\,[H(z; H_0, \theta_0) - H_0] \qquad (5.2)$$

Here $H(z) = H(z; H_0, \theta_0)$ Is the altitude corresponding to the current value of z. If $H(z; H_0, \theta_0) < H_0$, then $L(z; H_0, \theta_0; 0) < 0$. According to the meaning of the quantity $\mu(z; H_0, \theta_0; 0)$ and to the bound (4.4) this signifies that the very fact of the craft's descent in the planetary atmosphere renders a stabilizing effect which consists in a lessening of the perturbations of the trajectories of system (5.1), caused by the perturbations of its initial state (H_0, θ_0). From (5.2) it follows that this stabilizing effect is the greater, the greater is the density gradient λ, whereas the degree of lability of system (5.1), $L(z; H_0, \theta_0; 0)$ does not depend upon the form of the trajectories of system (5.1) but is determined solely by altitude difference $H(z; H_0, \theta_0) - H_0$. For example, let $H(z; H_0, \theta_0) - H_0 = 80$ km. For a descent onto the Earth, $\lambda \approx (1/7170)\,m^{-1}$. In this case $L(z; H_0, \theta_0; 0) \approx 11.16$, and by formula (4.4) for $n = 2$ we have $\mu(z; H_0, \theta_0; 0) \geqslant 235$. Returning to (1.2), (4.4) and (5.1), we can note that there exists an additional possibility of influencing the quantity $L(z; H_0, \theta_0; 0)$ and the lower bound for $\mu(z; H_0, \theta_0; 0)$, which consists of using as control functions the lift-drag ratio c_y / c_x and the drag coefficient c_x (*) as functions of H and θ.

6. Mathematical models of control processes often are given by equations of form (1.1) in which

$$X(t, x) = F(t, x, u(t)) \qquad (6.1)$$

Here $F(t, x, u)$ is a function of t, x, u, $u = (u_1, \ldots, u_r)$ is a vector in a real r dimensional linear normed space R_u^r, $u(t)$ is the control (the control function).

When experimenting with a concrete material object it is often required to bring out new properties of this object, different from the properties of its model (1.1), (6.1), as well as to define more precisely the boundary and the region of applicability of this model. By forcing the growth of the degree of lability of system (1.1), (6.1) with the aid of specially selected controls and by using precisely these controls when experimenting with the concrete material object, we can hope to establish in the quickest way the difference between the behavior of this object and of its model.

Let each of the controls $u = u(t)$, $t \geqslant t_0$, belonging to a certain admissible control set $U = \{u(t), t \geqslant t_0\}$, ensure the fulfillment of all the conditions of Theorem

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*) Editor's Note. English symbols for the lift-drag ratio, drag coefficient and lift coefficient are as follows: C_L/C_D, C_D and C_L, respectively.

1 on the function $X(t, x)$ excepting condition (3.1) which we do not take into accout here. Among the controls in U we can choose a control such that the system (1.1), (6.1) achieves a given magnitude of the degree of lability $L(t_1, x_0, t_0) = B$, where B is a specified number, in the minimal time $t_+ - t_0$, where

$$t_+ = \min_{u(t)\in U} t_1 \qquad (6.2)$$

It is obvious that the desired control $u = u_0(t),\ t_0 \leqslant t < t_+$ can be found from the functional equation

$$\int_{t_0}^{\min_{u(t)\in U} t_1} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} X_i(\tau, f(\tau, x_0, t_0))\, d\tau = B \qquad (6.3)$$

Here the function $x(t) = f(t, x_0, t_0),\ t_0 \leqslant t < t_+,\ x_0 \in V_0$ satisfies system (1.1), (6.1) in the presence of the given $x_0 \in V_0$ and of the desired $u = u_0(t),\ t_0 \leqslant t < t_+$.

All the preceding discussions can be extended to the case of piecewise-smooth controls by the method of matching.

If the values of the control function belong to a compact closure in R_u^r, the solution of the functional equation (6.2) can be reduced to the known time optimal problem [10] with an additional integral constraint of the isoperimetric type. Indeed, the solution of problem (6.2) is equivalent to the problem of minimizing the integral

$$t_1 = \int_{t_0}^{t_1} d\tau + t_0$$

subject to two constraints: the differential one (1.1), (6.1) (under the condition that the initial state $x(t_0) = x_0$ belongs to set V_0, while the control $u = u_0(t),\ t \geqslant t_0$, belongs to set U) and the integral one (the isoperimetric condition)

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} X_i(\tau, f(\tau, x_0, t_0))\, d\tau = B$$

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SOME EXACT SOLUTIONS OF A SYSTEM OF EQUATIONS OF ELECTROHYDRODYNAMICS

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A class of exact solutions of a system of equations of electrohydrodynamics is studied for which the electric current is directed along streamlines of the hydrodynamic flow. In the two-dimensional case the solution is written down explicitly. It is shown how to construct other exact solutions for which the collinearity condition of the electric current density and velocity vectors has not been satisfied, by using the solutions obtained. as an illustration, an exact solution for the flow of a unipolarly charged fluid in a channel with electrode-walls is constructed. It is shown that for a particular kind of hydrodynamic eddy current the solution of the two-dimensional system of equations can be reduced in some cases to finding the solution of a system of ordinary differential equations.

1. Let us examine the stationary flow of a unipolarly charged fluid. The parameter of the electrohydrodynamic interaction is assumed infinitesimal. A hydrodynamic stream of ideal incompressible homogeneous fluid has the potential $V^* = -\operatorname{grad} \Phi^*$. Ohm's law has the form $j^* = q^* (V^* + bE^*)$, where $b = \text{const}$ is the mobility. Let us introduce dimensionless quantities by means of formulas

$$x = l\xi,\ y = l\eta,\ z = l\zeta,\ \varphi^* = \frac{u_0 l}{b}\varphi,\ \Phi^* = u_0 l\Phi$$

$$q^* = \frac{\varepsilon_0 u_0}{4\pi b l} q \tag{1.1}$$

where φ^* is the electric field potential $E^* = -\operatorname{grad} \varphi^*$. Using the potentiality of the electric field and the velocity field, let us introduce the total potential $\chi = \Phi + \varphi$. The equations of electrohydrodynamics are [1]

$$\Delta\Phi = 0, \quad \operatorname{div}(q \operatorname{grad} \chi) = 0, \quad \Delta\chi = -q \tag{1.2}$$